J. Fluid Mech. (1971), *vol.* 50, *part* 1, *pp.* 177–188
Printed in Great Britain

An Oseen model of the two-dimensional flow of a stratified fluid over an obstacle

By KATHLEEN TRUSTRUM
School of Mathematical and Physical Sciences, University of Sussex

(Received 2 February 1970 and in revised form 24 February 1971)

The Oseen equations for the two-dimensional flow of a Boussinesq fluid over a thin barrier placed in a channel of finite depth are solved in the double limit $\nu \to 0$, $t \to \infty$ under the hypothesis that the velocity at the tip of the barrier is as weakly singular as possible. The predicted flow patterns and drag coefficients are in closer agreement with Davis's experimental observations than those of the Long model.

1. Introduction

Several authors, notably Long (1955), Yih (1960), Drazin & Moore (1967) and Miles (1968) have solved the steady state equations for the two-dimensional flow of a stratified fluid over an obstacle, using the Long model for which the governing equation is the reduced wave or Helmholtz's equation. The predicted flows provide a reasonable description of the real flow for small stratification and sufficiently small obstacles, but for larger stratification and larger obstacles the predicted flows contain complex lee-wave patterns with several closed streamlines (cf. Drazin & Moore 1967), which bear no resemblance to the observations of Davis (1969).

In an earlier paper (Trustrum 1964), the author solved the non-steady Oseen equations for the two-dimensional flow of a Boussinesq fluid† in a channel of finite depth past a distribution of sources and found that in the limit of infinite time, upstream influence can occur provided the inverse Froude number is greater than one. A mechanism is provided by internal gravity waves of zero frequency and finite group velocity (cf. Bretherton 1967). If upstream influence occurs, then the Long model is invalidated, though it may still provide a close approximation to the flow under certain conditions. Since upstream influence has been observed in stratified fluids by Long (1955) and in rotating fluids by Pritchard (1969), the aim of the present work was to investigate the flow over a thin vertical barrier, by means of a model which allowed upstream influence to occur, to see if the predicted flow patterns provided a satisfactory description of the real flows observed by Davis (1969).

For a given Fourier component in the vertical direction, the Oseen solution in the limit of infinite time has one unknown constant in the upstream flow and three in the downstream flow (cf. Trustrum 1964). The inviscid boundary conditions on the barrier are sufficient to determine the upstream flow but are

† A Boussinesq fluid is a fluid for which the Boussinesq approximation holds.

insufficient for the downstream flow. However, the downstream flow can be made determinate by following a suggestion of Stewartson (1968) and using the viscous boundary conditions on the downstream side of the barrier. This technique is used in this paper and the results, which are presented in figures 1 to 6, are encouraging.

2. Equations of motion

We consider the two-dimensional flow of an inviscid Boussinesq fluid past a vertical thin barrier of height $\frac{1}{4}H$ placed on the bottom of a channel of depth H, whose boundaries are rigid horizontal planes. At an infinite distance upstream of the barrier, the fluid has, at any finite time, a uniform velocity U parallel to the horizontal axis Ox and density $\rho_{-\infty}(y) = \rho_0 - (\rho_0 \beta H y/\pi)$, where y is the dimensionless vertical co-ordinate. The axes are chosen so that the origin is fixed at the foot of the barrier and the velocity $\mathbf{u} = (U + Uu, Uv)$ corresponds to the Cartesian co-ordinates $(Hx/\pi, Hy/\pi)$. On scaling the time by $H/\pi U$ and writing the density as $\rho_{-\infty}(y) + (U^2\pi/gH)\rho_0\rho$ the Oseen equations, in which squares and products of u, v and ρ are neglected, are

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x},$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = -\rho - \frac{\partial P}{\partial y},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} - k^2 v = 0,$$

where $\rho_0 U^2 P = \int g\rho_{-\infty}(y)\,dy + p$ and p is the pressure. A perturbation stream function ψ is defined from the continuity equation by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x,$$

and it follows that ψ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 \nabla^2\psi + k^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{2.1}$$

where $k^2 = g\beta H^2/\pi^2 U^2$ and $1/k$ is an internal Froude number.

On satisfying the conditions

$$\psi = 0 \quad \text{on} \quad y = 0, \pi,$$

on the boundaries of the channel, the formal solution of (2.1) in the limit as $t \to \infty$ for fixed (x, y), is (cf. Trustrum 1964)

$$\left.\begin{aligned} \psi &= \sum_{n=1}^{K} C_n \sin ny + \sum_{K+1}^{\infty} D_n e^{\alpha_n x} \sin ny \quad (x < 0) \\ \text{and} \qquad \psi &= \sum_{n=1}^{K} (A_n \cos \alpha_n x + D_n \sin \alpha_n x + B_n) \sin ny \\ &\quad + \sum_{K+1}^{\infty} (A_n e^{-\alpha_n x} + B_n + C_n) \sin ny \quad (x > 0), \end{aligned}\right\} \tag{2.2}$$

where $\alpha_n = |k^2 - n^2|^{\frac{1}{2}}$ and $K < k < K+1$. The terms independent of x represent internal gravity waves of zero frequency and finite group velocity whose energy, with respect to the chosen frame of reference, travels with speeds $U(1 \pm k/n)$ along Ox. Hence such waves can only propagate upstream if $n < k$ (cf. Bretherton 1967). The respective solutions for the pressure and density perturbations, P and ρ, are

$$\left.\begin{aligned} P &= -k\sum_{n=1}^{K} C_n \cos ny - \sum_{K+1}^{\infty} nD_n e^{\alpha_n x}\cos ny \quad (x<0),\\ P &= -\sum_{n=1}^{K} (nA_n \cos\alpha_n x + nD_n \sin\alpha_n x - kB_n)\cos ny\\ &\quad -\sum_{K+1}^{\infty} (nA_n e^{-\alpha_n x} - kB_n + kC_n)\cos ny \quad (x>0),\end{aligned}\right\} \tag{2.3}$$

and

$$\left.\begin{aligned} \rho &= -k\sum_{n=1}^{K} nC_n \sin ny - k^2\sum_{K+1}^{\infty} D_n e^{\alpha_n x}\sin ny \quad (x<0),\\ \rho &= -\sum_{n=1}^{K} (k^2A_n \cos\alpha_n x + k^2D_n \sin\alpha_n x - knB_n)\sin ny\\ &\quad -\sum_{K+1}^{\infty} (k^2A_n e^{-\alpha_n x} - knB_n + knC_n)\sin ny \quad (x>0).\end{aligned}\right\} \tag{2.4}$$

It follows from (2.2) that only one boundary condition is required on $x = 0$ to determine the upstream flow uniquely, whereas three boundary conditions are required for the downstream solution. Stewartson (1968) considered the double limit $\nu \to 0$ then $t \to \infty$, where ν is the kinematic viscosity for the analogous rotating fluid problem. He showed that a boundary layer occurs on the upstream side of the obstacle, but on the downstream side, the vorticity created by the no-slip condition is convected away by the fluid. Thus he concludes that the appropriate boundary conditions to describe a physically reasonable flow are the inviscid conditions on the upstream side of the obstacle and the viscous conditions on the downstream side. Applying these to the flow past a vertical thin barrier of dimensionless height $\frac{1}{4}\pi$, we obtain the following conditions:

$$\left.\begin{aligned} &\psi \text{ continuous across } x=0 \quad \text{for} \quad 0 \leqslant y \leqslant \pi,\\ &\partial\psi/\partial x,\ \rho,\ \partial P/\partial y \text{ continuous across } x=0 \quad \text{for} \quad \tfrac{1}{4}\pi \leqslant y \leqslant \pi,\end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} \psi(0,y) &= -y \quad &&\text{for} \quad 0 \leqslant y \leqslant \tfrac{1}{4}\pi,\\ (\partial\psi/\partial x)_{x=0+} &= 0 &&\text{for} \quad 0 \leqslant y \leqslant \tfrac{1}{4}\pi,\\ \rho(0+,y) &= 0 &&\text{for} \quad 0 \leqslant y \leqslant \tfrac{1}{4}\pi,\end{aligned}\right\} \tag{2.6}$$

where $x = 0+$ denotes the downstream side of the barrier. The boundary condition on $x = 0+$ for ρ assumes that the barrier is a conductor held at the undisturbed temperature and corresponds, in rotating fluid problems, to the body rotating with the angular velocity of the undisturbed fluid. We shall subsequently refer to this model as 'Oseen model A'.

We now write, more generally,

$$\left.\begin{aligned} &\left[\frac{\partial\psi}{\partial x}\right]_{x=0-}^{x=0+} = -g(y) = -\sum_{n=1}^{\infty} g_n \sin ny, \quad \left[\frac{\partial P}{\partial y}\right]_{x=0-}^{x=0+} = h(y) = \sum_{1}^{\infty} h_n \sin ny,\\ &\text{and} \qquad [\rho]_{x=0-}^{x=0+} = q(y) = \sum_{n=1}^{\infty} q_n \sin ny \quad \text{for} \quad 0 \leqslant y \leqslant \pi,\end{aligned}\right\} \tag{2.7}$$

where $g(y)$, $h(y)$ and $q(y)$ are zero for $y \geqslant \frac{1}{4}\pi$ to satisfy (2.5). On substituting (2.2) to (2.4) in (2.7) and equating the coefficients of $\sin ny$, we obtain

$$\left.\begin{aligned}
\psi &= \sum_{n=1}^{K} \frac{kh_n + nq_n}{2kn(n-k)} \sin ny + \sum_{K+1}^{\infty} \left[\frac{g_n}{2(n^2-k^2)^{\frac{1}{2}}} - \frac{h_n+q_n}{2(n^2-k^2)}\right] e^{\alpha_n x} \sin ny \quad (x<0),\\
\psi &= \sum_{n=1}^{K} \left[\frac{h_n+q_n}{n^2-k^2}\cos\alpha_n x - \frac{g_n}{(k^2-n^2)^{\frac{1}{2}}}\sin\alpha_n x - \frac{kh_n - nq_n}{2kn(n+k)}\right]\sin ny\\
&\quad + \sum_{K+1}^{\infty}\left[\left(\frac{g_n}{2(n^2-k^2)^{\frac{1}{2}}} + \frac{h_n+q_n}{2(n^2-k^2)}\right)e^{-\alpha_n x} - \frac{h_n+q_n}{n^2-k^2}\right]\sin ny \quad (x>0).
\end{aligned}\right\} \tag{2.8}$$

This expression for ψ is continuous across $x = 0$ and does not depend upon the Stewartson boundary conditions (2.6) but only on the relations (2.7).

It is to be noted that $g(y)$, $q(y)$ and $h(y)+q(y)$ give the discontinuities in vertical velocity, density and vorticity respectively, across the barrier. For the steady flow of a Boussinesq fluid, the density is a function of the stream function Ψ, which satisfies
$$\nabla^2\Psi + (g/\rho_0)\, d\rho/d\Psi = F(\Psi).$$
Since $\Psi = 0$ on the barrier, the density and vorticity are continuous across the barrier for arbitrary upstream conditions. Hence if we require the density and vorticity to be continuous across the barrier, $q(y)$ and $h(y)$ are zero and the Oseen model predicts no upstream influence, as the terms independent of x disappear in (2.8). This suggests that in the absence of viscosity, upstream influence is at most of the second order in the wave amplitude.

Returning to the solution of Oseen model A, we substitute (2.8) and a similar expression for ρ into (2.6) to give the conditions

$$\left.\begin{aligned}
&\sum_{=1}^{K} \frac{kh_n + nq_n}{2kn(k-n)}\sin ny + \sum_{K+1}^{\infty}\left[\frac{h_n+q_n}{2(n^2-k^2)} - \frac{g_n}{2(n^2-k^2)^{\frac{1}{2}}}\right]\sin ny = y \quad (0 \leqslant y \leqslant \tfrac{1}{4}\pi),\\
&\sum_{n=1}^{K} g_n \sin ny + \sum_{K+1}^{\infty}\left[\frac{h_n+q_n}{2(n^2-k^2)^{\frac{1}{2}}} + \frac{g_n}{2}\right]\sin ny = 0 \quad (0 \leqslant y \leqslant \tfrac{1}{4}\pi),\\
&\sum_{n=1}^{K} \frac{kh_n + (2k-n)q_n}{2(k-n)}\sin ny + \sum_{K+1}^{\infty}\left[\frac{k^2h_n + (2n^2-k^2)q_n}{2(n^2-k^2)} - \frac{k^2 g_n}{2(n^2-k^2)^{\frac{1}{2}}}\right]\sin ny = 0\\
&\qquad\qquad (0 \leqslant y \leqslant \tfrac{1}{4}\pi),
\end{aligned}\right\} \tag{2.9}$$

which are equivalent to three simultaneous linear integral equations for the determination of $g(y)$, $h(y)$ and $q(y)$ in the range $0 < y < \frac{1}{4}\pi$, where they are non-zero.

Since the functions $g(y)$, $h(y)$ and $q(y)$ are likely to be discontinuous at $y = \frac{1}{4}\pi$, the solution (2.8) will exhibit singularities on $y = \frac{1}{4}\pi$. This can be most easily seen by considering the solution for $k = 0$, in which case $\rho = 0$ and the perturbation stream function ψ is given by

$$\left.\begin{aligned}
\psi &= \sum_{1}^{\infty}\left[\frac{g_n}{2n} - \frac{h_n}{2n^2}\right]e^{nx}\sin ny \quad (x<0),\\
\psi &= \sum_{1}^{\infty}\left[\left(\frac{g_n}{2n} + \frac{h_n}{2n^2}\right)e^{-nx} - \frac{h_n}{n^2}\right]\sin ny \quad (x>0),
\end{aligned}\right\} \tag{2.10}$$

where g_n and h_n satisfy the first two equations of (2.9) with $k = 0$ and $q_n = 0$.

The upstream solution is irrotational, whereas the downstream solution is the superposition of an irrotational solution and a rotational solution, ψ_r, where

$$\psi_r = -\sum_1^\infty (h_n/n^2)\sin ny \quad \text{and} \quad \nabla^2\psi_r = \sum_1^\infty h_n \sin ny = h(y).$$

Thus the rotational part of the solution is independent of x and has zero vorticity for $y > \frac{1}{4}\pi$, that is outside the wake. So for $y > \frac{1}{4}\pi$ the effect of ψ_r is to superpose a uniform velocity on the flow to satisfy the conservation of mass. The above properties of the solution agree with the theoretical work of Stewartson (1956) on the solutions of the viscous Oseen equations as $\nu \to 0$.

However, the author has been unable to determine the precise nature of the singularity in $h(y)$ and $g(y)$ at $y = \frac{1}{4}\pi$ and as pointed out by a referee, the Oseen problem, as posed, admits an infinity of solutions. A consideration of the viscous Oseen equations in the limit as $\nu \to 0$ suggests that $\partial\psi/\partial y$ will have a simple discontinuity across $y = \frac{1}{4}\pi$ and that $h(y)$ will behave like $\delta(\frac{1}{4}\pi - y)$. This means that the Fourier series for $h(y)$, although defined, will not converge, but such Fourier series can be integrated term by term so that the expression (2.10) for ψ is convergent. Hence to restore uniqueness, we hypothesize that the behaviour of the solution near $y = \frac{1}{4}\pi$ is as weakly singular as possible. This particular solution is almost certainly picked out by the method described in §3.

For $k > 0$, the terms independent of x in the downstream solution will give rise to a similar wake, but for $k > 1$ the wake will be obscured by the lee-waves.

3. Method of solution

Equations (2.9) are solved by adapting the technique used by Drazin & Moore (1967) to solve the Long model of the same problem. The method is described by Jones (1964, p. 269) and expands $g(y)$, $h(y)$, $q(y)$, y and $\sin ny$ in terms of the complete set of orthogonal functions $\{\sin 4ty\}$ on the range $[0, \frac{1}{4}\pi]$ as follows:

$$\left.\begin{aligned} & g(y) = \sum_{t=1}^\infty G_t \sin 4ty, \quad h(y) = \sum_{t=1}^\infty H_t \sin 4ty, \quad q(y) = \sum_{t=1}^\infty Q_t \sin 4ty, \\ & y = \frac{8}{\pi}\sum_{s=1}^\infty Y_s \sin 4sy \quad \text{and} \quad \sin ny = \frac{8}{\pi}\sum_{s=1}^\infty P_{ns}\sin 4sy, \quad \text{where} \quad Y_s = (-1)^{s+1}\frac{\pi}{16s}, \\ & P_{ns} = \frac{(-1)^s 4s \sin\frac{1}{4}n\pi}{n^2 - 16s^2} \quad \text{for} \quad n \neq 4s \quad \text{and} \quad P_{ns} = \tfrac{1}{8}\pi \quad \text{for} \quad n = 4s. \end{aligned}\right\} \tag{3.1}$$

From (2.7) and (3.1)

$$\left.\begin{aligned} g_n &= \frac{2}{\pi}\int_0^{\frac{1}{4}\pi} g(y)\sin ny\, dy = \frac{2}{\pi}\sum_{t=1}^\infty G_t P_{nt}, \\ h_n &= \frac{2}{\pi}\sum_{t=1}^\infty H_t P_{nt} \quad \text{and} \quad q_n = \frac{2}{\pi}\sum_{t=1}^\infty Q_t P_{nt}. \end{aligned}\right\} \tag{3.2}$$

On substituting (3.1) and (3.2) into (2.9) and equating the coefficients of $\sin 4sy$, we obtain the following infinite system of equations, after some rearrangement:

$$\left.\begin{aligned} \sum_{t=1}^{\infty} (A_{st}H_t + B_{st}Q_t - C_{st}G_t) &= \tfrac{1}{2}\pi Y_s, \\ \sum_{t=1}^{\infty} (C_{st}H_t + C_{st}Q_t + D_{st}G_t) &= 0, \\ \sum_{t=1}^{\infty} (E_{st}H_t - F_{st}Q_t) &= \tfrac{1}{2}\pi Y_s, \end{aligned}\right\} \tag{3.3}$$

for $s = 1, 2, \ldots$, where

$$A_{st} = \sum_{n=1}^{K} \frac{P_{ns}P_{nt}}{2n(k-n)} + \sum_{K+1}^{\infty} \frac{P_{ns}P_{nt}}{2(n^2-k^2)}, \quad B_{st} = \sum_{n=1}^{K} \frac{P_{ns}P_{nt}}{2k(k-n)} + \sum_{K+1}^{\infty} \frac{P_{ns}P_{nt}}{2(n^2-k^2)},$$

$$C_{st} = \sum_{n=K+1}^{\infty} \frac{P_{ns}P_{nt}}{2(n^2-k^2)^{\frac{1}{2}}}, \quad D_{st} = \sum_{n=1}^{K} P_{ns}P_{nt} + \sum_{K+1}^{\infty} \frac{P_{ns}P_{nt}}{2},$$

$$E_{st} = \sum_{n=1}^{K} \frac{P_{ns}P_{nt}}{2kn}, \quad F_{st} = \sum_{n=1}^{K} \frac{P_{ns}P_{nt}}{2k^2} + \sum_{K+1}^{\infty} \frac{P_{ns}P_{nt}}{k^2}.$$

To find approximate solutions of (3.3), we assume G_t, H_t and Q_t are zero for $t > M$ and that P_{ns} is zero for $n > 6M$, where $M = 20$ for most of the results presented in this paper. The values of g_n, h_n and q_n are then calculated from (3.2) which in turn give the perturbation stream function ψ on substitution in (2.8).

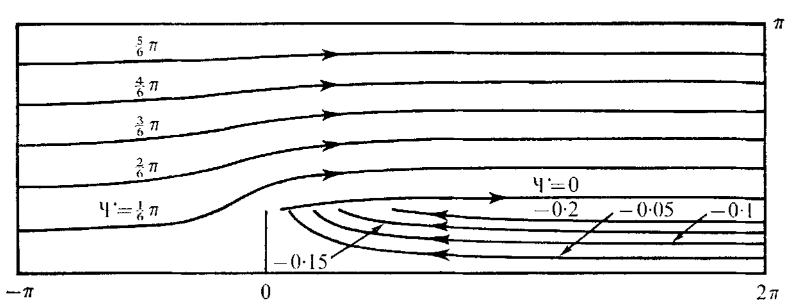


FIGURE 1. Calculated flow for $k = 0$ using Oseen model.

The drag coefficient is calculated from the formula

$$C_D = (\tfrac{1}{2}\rho_0 U^2 \tfrac{1}{4}H)^{-1} \int_0^{\frac{1}{4}\pi} \Big[p\Big]_{x=0+}^{x=0-} \frac{H}{\pi}\, dy,$$

which gives on using (2.7), (3.1) and truncating as above

$$C_D = \frac{8}{\pi}\int_0^{\frac{1}{4}\pi} y h(y)\, dy = \frac{1}{2}\sum_{t=1}^{M} (-1)^{t+1}\frac{H_t}{t}. \tag{3.4}$$

Following Davis (1969), the wave drag coefficient is given by

$$C_{DW} = 2\sum_{n=1}^{K} \left[g_n^2 + \frac{(h_n+q_n)^2}{k^2-n^2}\right].$$

The convergence of the numerical procedure was tested by calculating C_D for $M = 16$, 20 and 25 and fitting the formula

$$C_D(M) = a + bM^{-\gamma}$$

to the results. It was found that as k was increased from 0 to 4·5, γ varied between 0·5 and 0·8, with the convergence, in general, being faster for the higher values of k. The variation of $C_D(M)$ between $M = 16$ and $M = 25$ was less than 5 %.

In addition the technique was used to compute the solution (2.10) for the case $k = 0$ and the streamlines, shown in figure 1, are qualitatively similar to those obtained by Stewartson (1956) for the flow past a sphere. Although this solution had the slowest rate of convergence, the computed downstream flow for $y > \frac{1}{4}\pi$ closely approximates a uniform flow, as predicted by the theory. The results give one confidence in the numerical procedure for $k > 0$.

4. Discussion of results

Apart from the Oseen model A described above, two other Oseen models were also computed in which the third boundary condition of (2.6) was replaced as follows:

$$\rho(0+, y) = k^2 y \quad \text{for} \quad 0 \leqslant y \leqslant \tfrac{1}{4}\pi \quad \text{(Oseen model B)},$$

$$\rho \text{ continuous across } x = 0 \quad \text{for} \quad 0 \leqslant y \leqslant \tfrac{1}{4}\pi \quad \text{(Oseen model C)}.$$

Model B assumes that the fluid on the downstream side of the barrier originates from the bottom of the channel, where the density is the greatest, and is equivalent in rotating fluids to the obstacle being fixed. It is worth noting that the boundary condition $\partial\psi/\partial x = 0$ on the downstream side of the barrier implies that $\partial\rho/\partial x = 0$ there and so an insulating barrier leaves the problem indeterminate.

The flow patterns for the three models were almost identical in respect of the upstream velocity profile and the positions of the lee-wave crests and troughs, and the only significant variation was in the amplitudes of the lee-waves, which were largest for model B and smallest for model A. However, when the boundary condition on $\partial\psi/\partial x$ was made non-zero by putting $(\partial\psi/\partial x)_{x=0+} = y$ and $2y$, the upstream velocity profile and the positions of the crests and troughs were altered. In view of the similarities between the models A, B and C, only the flow patterns for C are reproduced.

The streamlines for the flow over a thin barrier for $k = 1·5$, calculated from the Oseen model C, are plotted in figure 2(*a*). These are to be compared with Davis's (1969) experimental observations shown in figure 2(*b*) and the flow calculated from the Long model shown in figure 2(*c*). The chief feature of figure 2(*a*) is that the positions of the first wave trough and crest are in much closer agreement with the observations than in the Long model though the amplitude is too small. It is also to be observed that the upstream flow has a slight shear and the velocity profile is similar to that used in linearized models of air flow over mountains. The flow pattern does not predict the isolated region of turbulence observed above the first trough, though the fluid is moving very slowly in this region. No details of the flow are given in the neighbourhood of the barrier, as the convergence of the solution is slower in this region and the results are less certain.

In figure 3(*a*), the flow pattern is shown for $k = 2{\cdot}25$ and is to be compared with Davis's observed and calculated flows, figures 3(*b*) and 3(*c*) respectively. Again the Oseen model bears more resemblance to the observed flow, with the positions of the first trough and crest being close to their observed positions. Also the Oseen model does not contain the closed streamlines of the Long model.

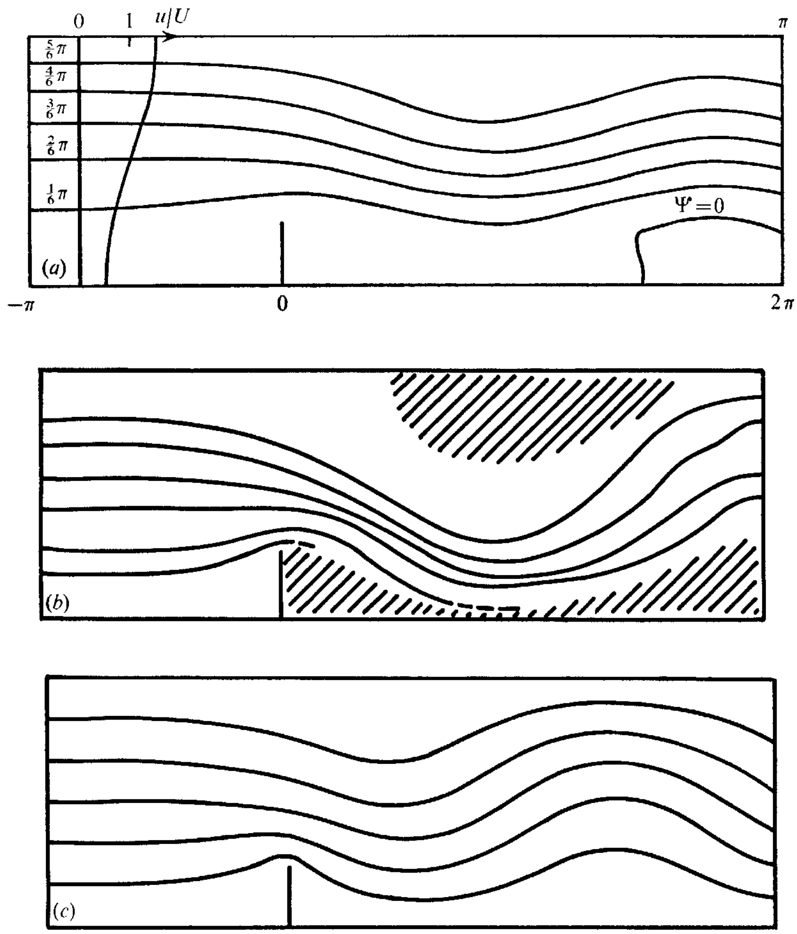


FIGURE 2. (*a*) Calculated flow for $k = 1{\cdot}5$ and upstream velocity profile, using Oseen model C. (*b*) Observed flow for $k = 1{\cdot}5$ (from Davis). (*c*) Calculated flow for $k = 1{\cdot}5$, using Long model (from Davis).

Figures 4(*a*) and 5 show the flow patterns for $k = 3{\cdot}6$ and $4{\cdot}5$, respectively, calculated from the Oseen model C and the observed flow for $k = 3{\cdot}6$ is shown in figure 4(*b*). The Long model flows for $k = 3{\cdot}5$ and $4{\cdot}5$, calculated by Drazin & Moore (1967), show complex lee-wave patterns extending throughout the channel, whereas the complex flow patterns in figures 4(*a*) and 5 are confined to the wake of the barrier. In fact the flows bear a reasonable resemblance to the following description of Davis (1969): "The flow remains laminar as it passes over the obstacle and forms an intense jet immediately downstream. Then this jet suddenly separates from the horizontal boundary behind the obstacle and erupts into a violently turbulent wake of approximately the same height as the

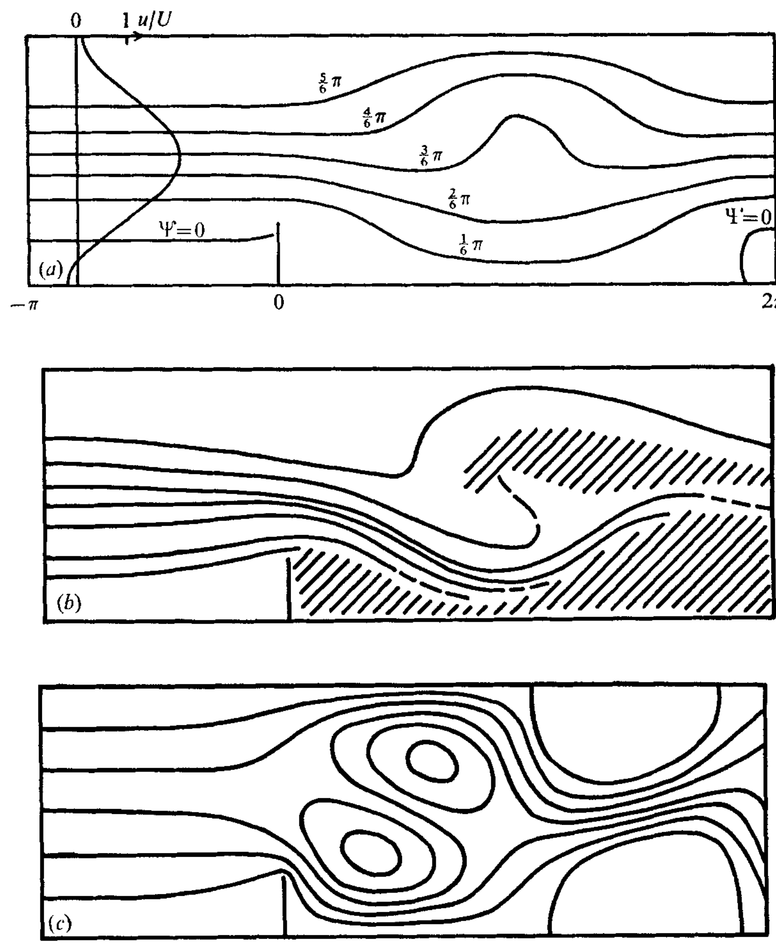
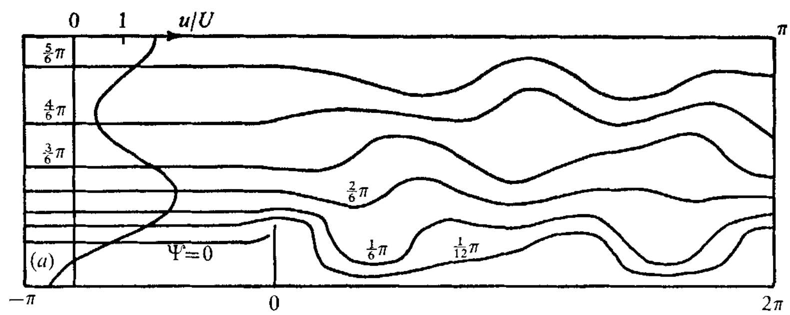


FIGURE 3. (*a*) Calculated flow for $k = 2{\cdot}25$ and upstream velocity profile, using Oseen model C. (*b*) Observed flow for $k = 2{\cdot}25$ (from Davis). (*c*) Calculated flow for $k = 2{\cdot}25$, using Long model (from Davis).

obstacle. Those streamlines which originate at height greater than the obstacle are not greatly disturbed by the barrier and in no part of the wake is there any evidence of organized wave motion."

However there are important differences in the upstream flow between the Oseen model and Davis's observations. The Oseen model for $k = 2{\cdot}25$, $3{\cdot}6$ and $4{\cdot}5$ predicts reversed flow ahead of the barrier, which would in practice be manifested as blocked flow. Also for $k = 3{\cdot}6$ and $4{\cdot}5$, the velocity profile shows a weak jet just above the top of the barrier. Such phenomena were observed by Long (1955), but Davis found no evidence of blocking or jet formation in his experiments, though he did find discrepancies between the theoretical (Long model) and observed position of the upstream streamlines. Pritchard (1969) says of his experiments with rotating fluids: "An interesting feature of these experiments is that the body may generate disturbances in which the particle velocities



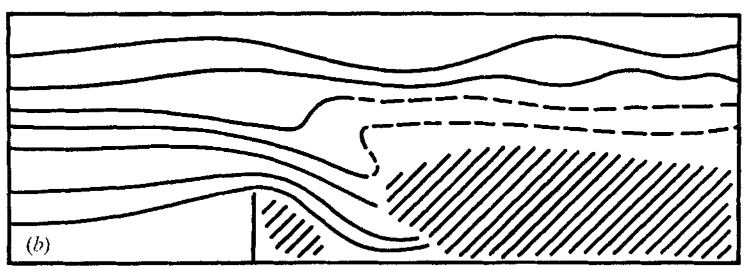


FIGURE 4. (a) Calculated flow for $k = 3{\cdot}6$ and upstream velocity profile, using Oseen model C. (b) Observed flow for $k = 3{\cdot}6$ (from Davis).

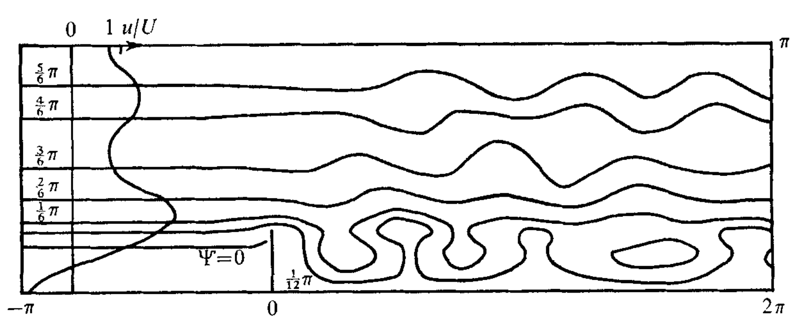


FIGURE 5. Calculated flow for $k = 4{\cdot}5$ and upstream velocity profile, using Oseen model C.

on the axis (ahead of the body) exceed the velocity of the body." This discrepancy between the results of the stratified and rotating fluid experiments might be explained by the different types of turbulence which occur in the two flows or by the different techniques used to observe upstream influence. Another feature of the Oseen model, which was pointed out by a referee, is that it predicts the fluid in the blocked region to be denser than any of the fluid at $t = 0$, so in this respect the model is unrealistic.

In figure 6, the drag coefficient C_D and wave drag coefficient C_{DW} are plotted against k for the Oseen models A, B and C and the experimental results of Davis (1969) are given for comparison. It should be noted that Davis measured the drag of a thin barrier of dimensionless height $\frac{1}{4}\pi$ with one edge at the height $\frac{1}{2}\pi$ and so his measurements are not for the observed flows shown in figures 2 (*b*), 3 (*b*)

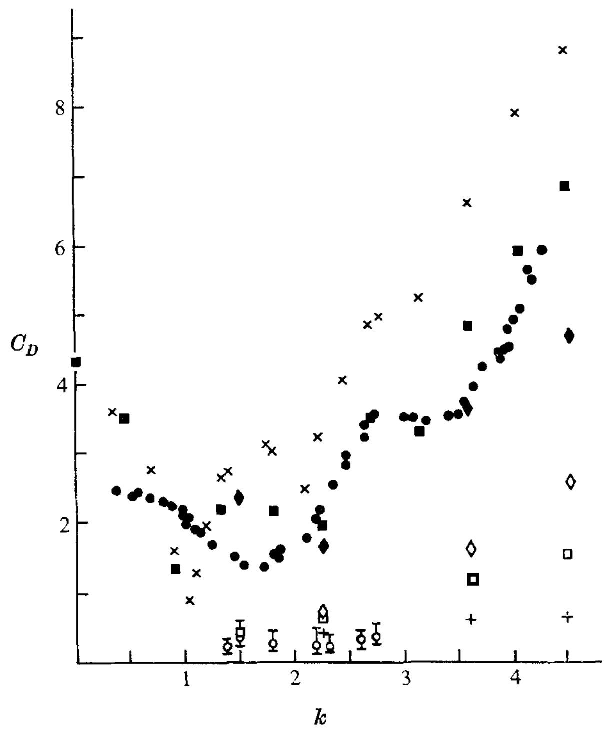


FIGURE 6. The drag coefficient C_D and wave drag coefficient C_{DW} for a thin barrier of height $\frac{1}{4}\pi$ placed on bottom of channel. $\times$, C_D values and $+$, C_{DW} values for Oseen model A: ◆, C_D values and ◇, C_{DW} values for Oseen model B: ■, C_D values and □, C_{DW} values for Oseen model C: ●, C_D values and ○, C_{DW} values measured for barrier with one edge at height $\frac{1}{2}\pi$ (from Davis).

and 4 (*b*). Although the boundary condition on ρ clearly has a significant effect on the magnitudes of C_D and C_{DW}, the behaviour of C_D as a function of k is qualitatively similar for the three Oseen models and the experiments, with C_D initially decreasing and finally increasing, almost linearly, with k. Maxworthy (1970), in his experiments on a sphere rising through a rotating cylinder of water, found that C_D initially decreased and finally increased linearly with the inverse Rossby number. The initial decrease of C_D as the inverse Rossby number increases from zero, was also predicted theoretically by Stewartson (1968). One curious feature of the results is that for $k > 2$, the Oseen model with the largest values of C_D has the smallest values of C_{DW} and vice versa.

5. Conclusions

The results presented in this paper support Stewartson's (1968) conjecture that the Oseen model gives "a qualitatively correct description of the grosser features of the flow", even though the approximation is invalid near the obstacle. In particular, for the stratified flow over a thin barrier it predicts the position but not the amplitude of the first wave crest reasonably accurately for $k = 1{\cdot}5$ and $2{\cdot}25$, which the Long model fails to do. For $k = 3{\cdot}6$ and $4{\cdot}5$ the downstream flow agrees fairly well with the descriptions of Davis (1969), with the flow pattern showing relatively small disturbances from uniform flow in the region above the wake whereas in the wake the streamlines are much more contorted. Also the predicted values of the drag coefficient for different values of k bear a qualitative resemblance to the values measured by Davis for a thin barrier placed with one edge in the middle of the channel. The most serious discrepancy with Davis's results is the difference in the upstream flows. The Oseen models all predict upstream influence for $k > 1$ and for the higher values of k the flow is blocked, whereas Davis did not detect any blocking, though possibly such phenomena are easier to detect for a body towed along the axis of a rotating fluid, where there is little interaction with the boundary layer.

The author would like to thank the referees for their helpful suggestions and criticisms. She would also like to acknowledge the work of Mr M. A. Myint on the Oseen model of the flow of a stratified fluid over an obstacle generated by point body forces and heat sources, which stimulated the present work.

REFERENCES

Bretherton, F. P. 1967 The time-dependent motion due to a cylinder moving in an unbounded rotating or stratified fluid. *J. Fluid Mech.* **28**, 545.

Davis, R. E. 1969 The two-dimensional flow of a stratified fluid over an obstacle. *J. Fluid Mech.* **36**, 127.

Drazin, P. G. & Moore, D. W. 1967 Steady two-dimensional flow of a fluid of variable density over an obstacle. *J. Fluid Mech.* **28**, 353.

Jones, D. S. 1964 *The Theory of Electromagnetism*. Pergamon.

Long, R. R. 1955 Some aspects of the flow of stratified fluids; III. Continuous density gradients. *Tellus*, **7**, 341.

Maxworthy, T. 1970 The flow created by a sphere moving along the axis of a rotating, slightly-viscous fluid. *J. Fluid Mech.* **40**, 453.

Miles, J. W. 1968 Lee waves in a stratified fluid, Part 1. Thin barrier. *J. Fluid Mech.* **32**, 549.

Pritchard, W. G. 1969 The motion generated by a body moving along the axis of a uniformly rotating fluid. *J. Fluid Mech.* **39**, 443.

Stewartson, K. 1956 On the steady flow past a sphere at high Reynolds number using Oseen's approximation. *Phil. Mag.* (8) **1**, 345.

Stewartson, K. 1968 On inviscid flow of a rotating fluid past an axially symmetric body using Oseen's equations. *Quart. J. Mech. Appl. Math.* **21**, 353.

Trustrum, K. 1964 Rotating and stratified fluid flow. *J. Fluid Mech.* **19**, 415.

Yih, C. S. 1960 Exact solutions for steady two-dimensional flow of a stratified fluid. *J. Fluid Mech.* **9**, 161.